J. Fluid Mech. (1983), *vol.* 133, *pp.* 287–296
Printed in Great Britain

On the calculation of separation bubbles

By TUNCER CEBECI
Mechanical Engineering Department, California State University, Long Beach
AND KEITH STEWARTSON
Department of Mathematics, University College London

(Received 2 August 1982 and in revised form 14 April 1983)

The interactive boundary-layer equations for a flat plate are solved numerically when the external velocity field is piecewise linear and would provoke separation if the response of the boundary layer were neglected. A comparable problem had already been solved by Briley using the full Navier–Stokes equations. The equations are solved for various values of the Reynolds number and x_0, a parameter defining the corner point of the external velocity. It is found that flows with a limited region of separation can be computed, but that, if x_0 is too large, the numerical procedure breaks down. Furthermore, this maximum value is a decreasing function of R and seems to approach the value 0.12 predicted by classical theory as $R \to \infty$. Comparison with Briley's results indicate a reasonable agreement except that different values of x_0 are appropriate. It is conjectured that, once x_0 increases above the acceptable maximum, rapid changes occur in the flow properties when R is large.

1. Introduction

Some ten years ago, Briley (1971) published an important paper on the calculation of separation bubbles using the Navier–Stokes equations. He considered the flow in a rectangle $\frac{1}{20} < x < \frac{1}{2}, 0 < y < \frac{3}{80} = y_e$ of which the side $y = 0$ is a fixed wall and on the side $y = y_e$ the velocity component in the x-direction is prescribed to be

$$\bar{u}_e = \begin{cases} 1-x & (x < x_0), \\ 1-x_0 & (x \geqslant x_0), \end{cases}$$

where x_0 is a constant typically about 0.2. The Reynolds number R of the flow is $10^6/48$ so that $y_e \approx 5.4R^{-\frac{1}{2}}$.

Were $R = \infty$, the problem would be closely related to the flow discussed by Howarth (1938), and separation would have been expected to occur at $x \approx 0.12$, according to classical boundary-layer theory, and to be accompanied by a singularity of the Goldstein type (see Goldstein 1948). The question Briley addressed is how far are the solutions of the boundary-layer equations relevant to the solutions of the full Navier–Stokes equations at large but finite R. To this end he assumed that the velocity and vorticity distributions at $x = \frac{1}{20}$ are given by Howarth's solution, that the vorticity is zero at $y = y_e$ and that the boundary-layer assumptions hold at the downstream end, $x = \frac{1}{2}$, of the rectangle.

Specific calculations were carried out for $x_0 = 0.157$, 0.170, 0.195 and 0.202, and the values of the skin friction and displacement thickness which he obtained are displayed in figures 1 and 2. It can be seen that Howarth's solution is well reproduced when $x - \frac{1}{20}$ is small, but wide divergences develop as the classical separation point $x_s = 0.12$ is approached, and separation, if indeed it occurs, does so without any hint

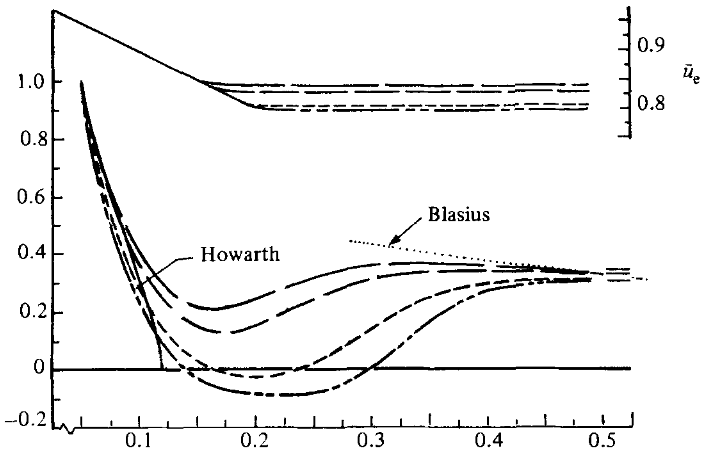

FIGURE 1. Values of reduced skin friction τ as functions of x for $R = 10^6/48$ and various x_0, taken from Briley (1971): ——, $x_0 = 0.157$; ---, 0.170; - - - -, 0.195; -–-, 0.202. The curve —— shows the values obtained by Howarth, and the curve ... shows the values from Blasius' formula with $x_0 = 0.202$.

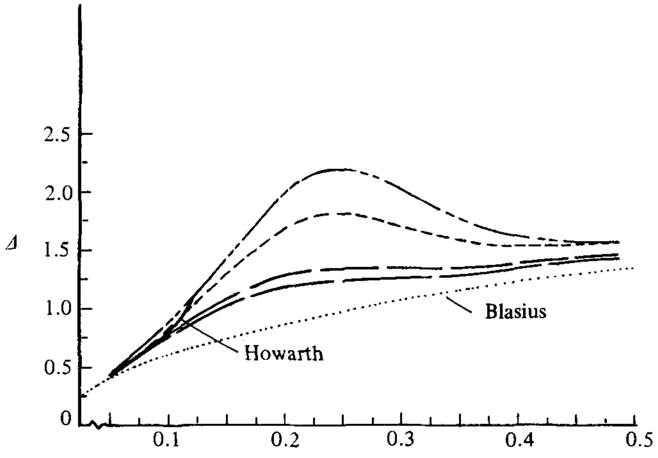

FIGURE 2. Values of reduced displacement thickness Δ taken from Briley (1971): ——, $x_0 = 0.157$; ---, 0.170; ----, 0.195; -–-, 0.202. The curve —— shows the values obtained by Howarth, and the curve ... shows the values from Blasius' formula with $x_0 = 0.202$.

of a singularity. The line $y = y_e$ is close to the outer edge of the boundary layer, but tests showed that increasing y_e would have little effect on the solution properties. More significantly, an attempt to obtain steady-state solutions at larger values of R failed owing to 'instability which is believed to be of physical origin' (Briley 1971).

Since then it has become abundantly clear that, while the classical two-dimensional boundary layer is of little practical value once separation occurs, it is relatively simple to modify the theory to a form which permits separation and reattachment to occur in a smooth way. The only requirement is that the external velocity should include not only a contribution from classical inviscid theory but also a contribution from the displacement thickness of the boundary layer. The mathematical justification rests on the theory of the triple deck, recently reviewed by Stewartson (1982) and

Smith (1982). The regularity of the solution at separation was demonstrated numerically by Dijkstra (1978) and Smith (1979).

A study was made by Cebeci, Stewartson & Williams (1980) of separation bubbles near the leading edge of a thin airfoil at large but finite values of R, using an interactive boundary-layer theory, and was in accord with these fundamental ideas. Little difficulty was experienced in computing the boundary layer when the bubble is small. Above a certain critical angle of attach α_c, however, the iteration procedure suddenly failed to converge, and it was conjectured that in fact no solutions exist of the type assumed, even though at lower angles of attack no peculiarity of the solutions had been developing. A related study using triple-deck theory (Stewartson, Smith & Kaups 1982) confirmed this conclusion, and, moreover, demonstrated that when separated solutions do exist they are not unique. Further support is provided by an earlier study (Stewartson 1970), which shows that a finite singularity cannot be removed using a local concept of a triple deck.

In this paper we apply interactive theory to Briley's problem and demonstrate that this modification to classical boundary-layer theory predicts flow properties in qualitative agreement with the solution of the full Navier–Stokes equations even at values of R as low as those considered by Briley. We shall also show that, as R increases, the greatest value of x_0 at which converged solutions can be found decreases, and that in all probability it approaches 0.12 as $R \to \infty$. Thus the Howarth boundary layer is not the limit solution of the interactive equations for any $x_0 > 0.12$.

2. Basic equations

The interaction theory requires us to solve the boundary-layer equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_e\frac{du_e}{dx}+\frac{\partial^2 u}{\partial Y^2} \tag{1b}$$

subject to the boundary conditions

$$u=v=0 \quad \text{at} \quad Y=0, \quad x>0 \tag{2a}$$

$$u(x,Y)\to u_e(x) \quad \text{as} \quad Y\to\infty, \tag{2b}$$

where

$$u_e(x)=\bar{u}_e(x)+\frac{1}{\pi R^{\frac{1}{2}}}\int_0^\infty \frac{\Delta'(x_1)}{x-x_1}\,dx_1, \tag{3}$$

$$\Delta(x)=\int_0^\infty (u_e-u)\,dY. \tag{4}$$

Here a prime denotes differentiation with respect to x. The variables y and Y are related by the formula $Y=yR^{\frac{1}{2}}$, so that in Briley's problem y_e corresponds to $Y_e=5.4$.

The solution procedure of the system given by (1)–(4) is similar to that described in Cebeci *et al.* (1980), which, in turn, is based on ideas first formulated by Veldman (1979). In this procedure, first $\Delta(x)$ is replaced by

$$\Delta(x)=\frac{1.7208(b+u_\infty^{\frac{1}{2}}x)\,x^{\frac{1}{2}}}{b+x}=\Delta_0(x) \tag{5}$$

in the Hilbert integral (3), where b is a constant finally chosen to be 0.3, and $u_\infty=1-x_0$. The correction term in (5) coincides with the displacement thickness due to a uniform velocity of unity when x/b is small, and due to a uniform velocity u_∞ when x/b is large. Hence $\Delta_0(x)$ has a finite derivative when $x = 0$, and Δ_0' is very small when x is large. To compensate, a term

$$\frac{1.7208b^{\frac{3}{2}}}{(b+x)^2R^{\frac{1}{2}}}(1-u_\infty^{\frac{1}{2}}) \tag{6}$$

is added to the right-hand side of (3). Secondly, the range of integration of the Hilbert integral is changed to $0.084 \leqslant x \leqslant 1.464$; the errors induced by this approximation are not thought to be serious. Thirdly, the boundary-layer equations are integrated over the range $0 \leqslant x \leqslant 0.072$ using similarity variables x and $Y/x^{\frac{1}{2}}$ and also with the neglect of the Hilbert integral. Thus the solution at $x = 0.072$ is fixed once and for all. Fourthly, the integration from $x = 0.084$ onwards is carried out in a series of sweeps. In each sweep the Hilbert integral is discretized and the Keller box scheme is used to integrate the equations. As the sweep proceeds, updated values of $\Delta_0(x)$ are used to compute $u_e(x)$ as soon as they have been generated. Fifthly, the FLARE approximation (see Cebeci *et al.* 1980; Reyhner & Flugge-Lotz 1968), in which $u\,\partial u/\partial x$ is neglected whenever $u < 0$, is used. Sixthly, over-relaxation is used to speed the convergence. As the sweeps progress, a weak instability develops near the switch from the standard form of integration to the inverse form at $x = 0.072$. This is removed by the seventh modification; namely to smooth the profiles at the three adjacent points $x = 0.072$, 0.084 and 0.096, by weighting them in the ratio $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and using the resultant as the starting profile at $x = 0.084$. Finally, to smooth the introduction of the Hilbert integral further, the Reynolds number is multipled by $\operatorname{cosec}[(x-x_1)\,\pi/2\Delta x]$ in the integration over the four points starting from $x_1 = 0.084$, where $\Delta x = 0.012$. A similar device is adopted for the last four points of the range of integration in x. A number of numerical experiments were carried out to test the efficacy of these modifications, and the results were generally satisfactory.

3. Results and discussion

Computations were carried out for three values of R, namely $R_1 = 10^6/144$, $R_2 = 10^6/48$ and $R_3 = 10^8/144$, of which the second is relevant to Briley's study, and we shall discuss the results for this case first. In figures 3, 4, 5 we display the variation of $\tau = (\partial u/\partial y)_{y=0}, \Delta$ and $\bar{u}_e(x)$ as functions of x over the range $0.084 \leqslant x \leqslant 1.2$ for $x_0 = 0.21$, 0.22, 0.23, 0.24. The properties of these functions for $x > 1.2$ are affected by truncation errors to a very small extent at $x = 1.212$ and significantly at the end of the range of integration, but it is not believed that these have a noticeable effect on the solution in $x \leqslant 1.2$. It can be seen that the behaviour of τ and Δ near $x = 0.084$ is close to that predicted by Howarth's theory, while, near $x = 1.2$, τ is well approximated by the Blasius solution corresponding to $u_e = 1-x_0$. The displacement thickness Δ/u_e, however, only agrees locally with that from the Blasius solution at $x = \frac{1}{2}$, and it is clear that further downstream some undershoot occurs.

Comparison with Briley's solution shows fairly good agreement except for the value of x_0. Thus the positions of separation and reattachment in Briley's calculation at $x_0 = 0.202$ are close to the present results at $x_0 = 0.23$. The two sets of graphs of Δ are also close as far as the peak. Thereafter, however, Briley's values show a marked decrease, whereas in the present studies Δ decreases quite slowly before rising again beyond the range of his data. A typical comparison is shown in figure 6. Both sets of data are well above the Blasius formula in $x < 0.5$, but the boundary-layer calculations finally fall just below it near $x = 1.2$. The different values of x_0 appearing

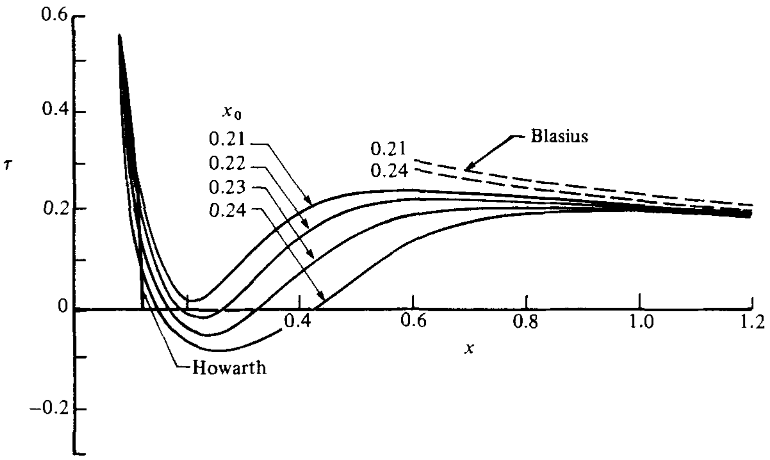

FIGURE 3. Values of reduced skin friction τ as functions of x for $R = 10^6/48$ and various x_0 using interactive boundary-layer theory. Corresponding results from Howarth and Blasius are added for comparison purposes.

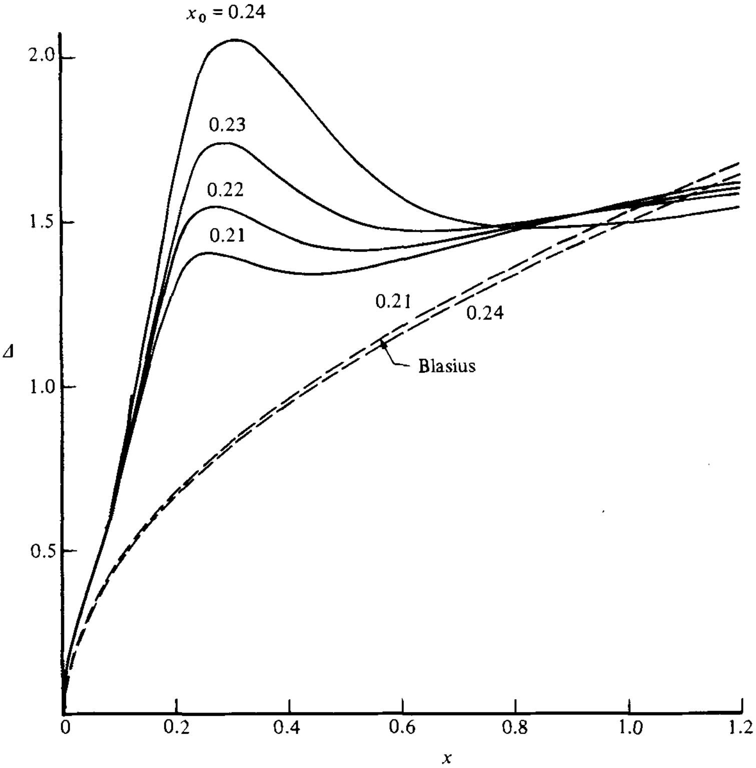

FIGURE 4. Values of Δ as functions of x for $R = 10^6/48$ and various x_0 using interactive boundary-layer theory.

in the two solutions may be due to the use of a FLARE approximation in the boundary-layer approach, and strictly it should be replaced by a DUIT procedure, which is an iterative procedure involving upstream integration as described by Williams (1975) and Cebeci, Keller & Williams (1979), or by a time-dependent scheme as employed by Cebeci (1983). However, the errors made in using FLARE are

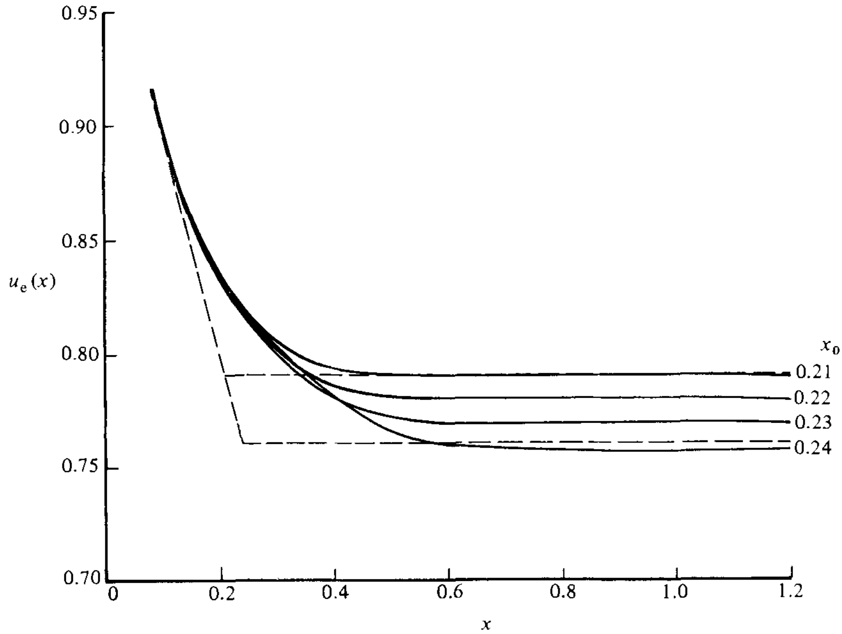

FIGURE 5. Variation of u_e (solid lines) and $\bar{u}_e$ (hatched lines) with x for various values of x_0 using interactive boundary-layer theory; $R = 10^6/48$.

not likely to be of a fundamental nature. In particular, the failure of the iterations to converge cannot be ascribed to a quirk of the method. For triple-deck theory, which is completely self-consistent, also only permits steady-state solutions if the domain of separation, after appropriate scaling, is limited in extent (Stewartson *et al.* 1982). Again, the forcing velocity $\bar{u}_e$ is used differently. Briley required $u = \bar{u}_e$ when $Y = Y_e$, whereas here $\bar{u}_e$ is the assumed basic inviscid velocity distribution from which the external velocity $u_e(x)$ is derived. If R were infinite this would mean that $u \rightarrow \bar{u}_e$ as $y \rightarrow \infty$, and $u \rightarrow u_e$ as $Y \rightarrow \infty$ and $y \rightarrow 0$ simultaneously; this condition is a necessary requirement for the validity of interactive theory. For finite R we assume that $u \approx u_e$ when $Y = yR^{\frac{1}{2}}$ is large and y is small. For example, if $R = 10^6/48$, the outer edge of the boundary layer may be taken as $Y \approx 7$, and the corresponding value of y is ≈ 0.05. Another minor source of error which might account for this difference is that, in both approaches, Howarth's solution is used as a boundary condition at the upstream end of the range of integration. Strictly, the interaction should disturb that solution, and in neither method are the final converged flow properties completely smooth there, once separation has occurred. Finally, we draw attention to the value Y_e of Y at which the outer boundary condition is applied in Briley's solution. He chose $Y_e = 5.4$. To be strictly equivalent to the calculations reported here, Y_e should be infinite, although difficulties connected with the assumed form of $\bar{u}_e$ might then arise because the Reynolds number is finite. Briley carried out some tests on this feature and concluded that there would be little change if Y_e were increased to 7.2, but this edge is still rather close to the outer edge of the boundary layer. Carter (1975) used his inverse boundary-layer technique to compute the flow properties taking one set of values of Δ found by Briley as prescribed. The agreement between his results and Briley's is good for all properties tested. This suggests that the principal discrepancies between the results of our approach and Briley's are due to the calculations of Δ. We conclude that the region of useful applicability of a Navier–Stokes approach, which strictly

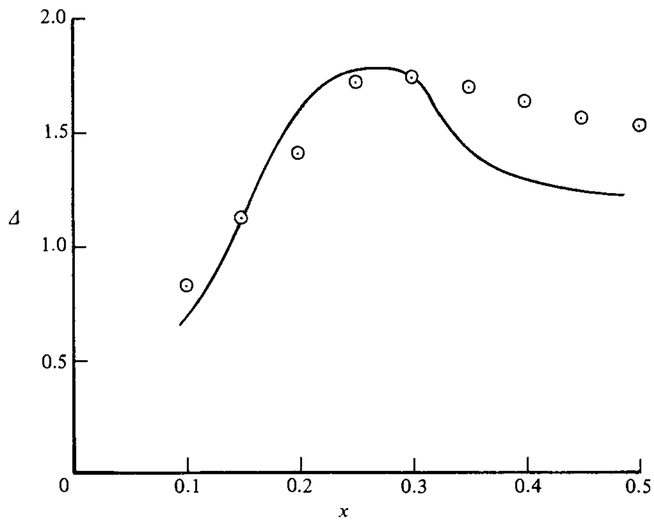

FIGURE 6. Comparison of the calculation for Δ for $R = 10^6/48$ by Briley (solid line) for $x_0 = 0.202$ and with the present method for $x_0 = 0.23$ (symbols).

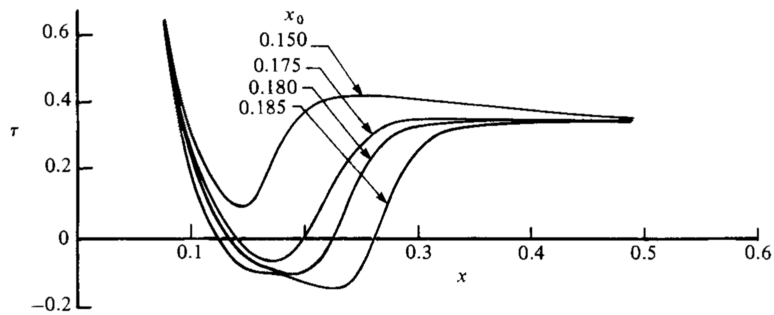

FIGURE 7. Variation of τ with x for various values of x_0 using interactive boundary-layer theory; $R = 10^8/144$.

must be bounded above in R, overlaps in the present problem with that of an interactive boundary-layer approach, which strictly must be bounded below in R. Hence reliable results can be obtained using one of the methods over the whole range of values of R.

Attempts were also made to obtain converged solutions at larger values of x_0. After forty sweeps at $x_0 = 0.26$, there were indications that eventually the solution would converge, but only after a further large number of sweeps were carried out. At $x_0 = 0.28$, rapid and violent oscillations developed in τ after a few sweeps, i.e. changes in τ of order unity occurred in adjacent stations of x and the solutions diverged.

Similar results were obtained at $R = R_1$ $(= 10^6/144)$, the main differences being that the critical value of x_0 to induce separation is reduced from $x_0 = 0.215$ at $R = R_2$ to $x_0 = 0.238$ at $R = R_1$ and the corresponding peak Δ is somewhat enhanced.

Finally, a study was carried out at $R = R_3$ $(=10^8/144)$ for $x_0 = 0.185, 0.180, 0.175$ and 0.150, and the variations of τ, Δ, u_e obtained are displayed in figures 7, 8 and 9. The critical value of x_0 increases to approximately 0.165 and the peak in Δ is enhanced. A study was also made at $x_0 = 0.1875$, but the separated region grew monotonically with the sweep number, and it is considered that the solution has failed to converge. In figure 10 we display the variation of the separation and reattachment

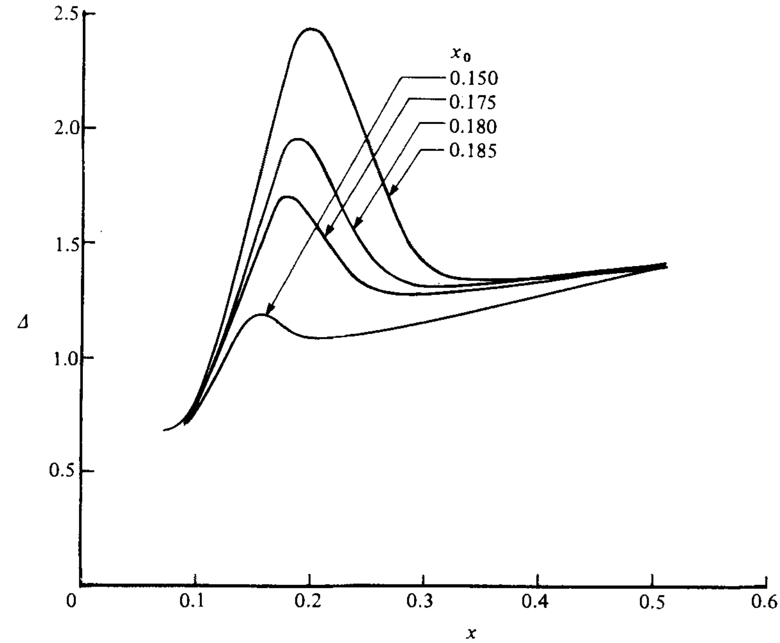

FIGURE 8. Variation of Δ with x for various values of x_0 using interactive boundary-layer theory; $R = 10^8/144$.

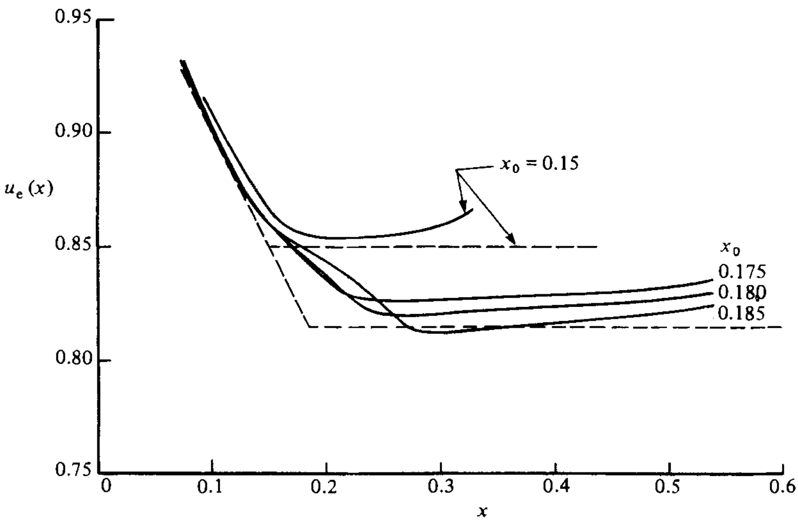

FIGURE 9. Variation of u_e (solid lines) and $\bar{u}_e$ (hatched lines) with x for various values of x_0 using interactive boundary-layer theory; $R = 10^8/144$.

points as functions of sweep number, and as contrast show the corresponding situation when $x_0 = 0.1825$.

It appears therefore that, for all values of R considered, there is an upper limit to the value of x_0 at which satisfactory converged solutions can be obtained, although the manner of breakdown varies depending on R. A similar phenomenon occurs in the leading-edge studies of Cebeci *et al.* (1980) and in the triple-deck theory of marginal separation (see Williams 1975). The critical value of x_0 for separation is a decreasing function of R and no doubt approaches 0.12 as $R \to \infty$ and Howarth's theory becomes relevant. This result is in agreement with the theory of marginal

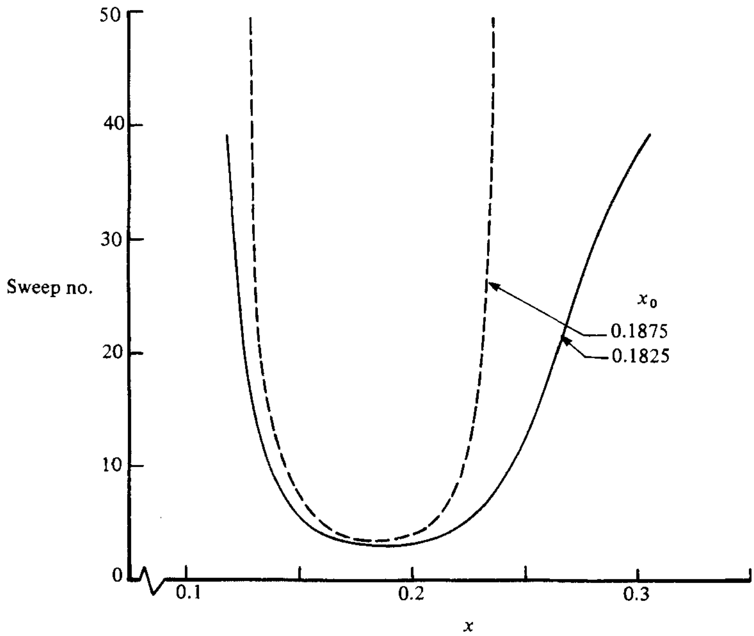

FIGURE 10. Length of the separated region as a function of sweep number for $R = 10^8/144$ and $x_0 = 0.1875$, 0.1825. The left-hand parts of the curve are the separation point and the right-hand parts the reattachment points.

separation, but $x_0 \rightarrow 0.12$ as $R \rightarrow \infty$ rather quicker than that theory would predict. It has proved too difficult at present to obtain satisfactory estimates to the upper bounds of x_0 at which solutions can be found, but it does appear that it also decreases with R, as the asymptotic theory requires.

Thus there is a bound to the usefulness of interactive boundary-layer theory, and, once it is exceeded, the theory in some sense goes sour. Possibly the global flow properties then rapidly change over to those corresponding to Kirchhoff free-streamline flow as discussed by Smith (1979). In this connection it is noted that Briley experienced insuperable difficulties in obtaining a converged solution at values of $x_0 \approx 0.2$ when he increased R by a factor of 9. One possible explanation is that, once the upper limit x_c of x_0 is exceeded, the region of separated flow predicted by the Navier–Stokes equation rapidly increases with $x_c - x_0$ and the range of values of x in his integration rectangle is inadequate to describe this phenomenon. In that event the elucidation of the flow properties in these conditions would be a problem of great importance at the present time.

This research was supported under the National Science Foundation Grant no. MEA-8018565. The authors are grateful to Dr W. R. Briley for helpful comments on an earlier version of this paper.

REFERENCES

BRILEY, W. R. 1971 A numerical study of laminar separation bubbles using the Navier–Stokes equations. *J. Fluid Mech.* **47**, 713–736.

CARTER, J. E. 1975 Inverse solutions for laminar boundary-layer flows with separation and reattachment. *NASA Tech. Rep.* TR R-447.

CEBECI, T. 1983 A time-dependent approach for calculating steady inverse boundary-layer flows with separation. *Proc. R. Soc. Lond.* (to be published).

CEBECI, T., KELLER, H. B. & WILLIAMS, P. G. 1979 Separating boundary-layer flow calculations. *J. Comp. Phys.* **31**, 363–378.

CEBECI, T., STEWARTSON, K. & WILLIAMS, P. G. 1980 Separation and reattachment near the leading edge of a thin airfoil at incidence. Computation of viscous–inviscid interactions. *Agard Conf. Proc.* no. 291, Colorado Springs.

DIJKSTRA, K. 1978 Separating incompressible laminar boundary-layer flow over a smooth step of small height. In *Proc. 6th Intl Conf. Numerical Methods in Fluid Dynamics, Tbilisi, USSR*, pp. 45–52.

GOLDSTEIN, S. 1948 On laminar boundary-layer flow near a point of separation. *Q. J. Mech. Appl. Maths* **1**, 43–69.

HOWARTH, L. 1938 On the solution of the laminar boundary-layer equation. *Proc. R. Soc. Lond.* **A164**, 547.

REYHNER, T. A. & FLUGGE-LOTZ, I. 1968 The interaction of a shock wave with a laminar boundary layer. *Intl. J. Nonlinear Mech.* **3**, 173–199.

SMITH, F. T. 1979 Laminar flow of an incompressible fluid past a bluff body, the separation, reattachment, eddy properties and drag. *J. Fluid Mech.* **92**, 171–205.

SMITH, F. T. 1982 Theory of laminar streaming flows. *IMA J. Mech.* **28**, 207–281.

STEWARTSON, K. 1970 Is the singularity at separation removable? *J. Fluid Mech.* **44**, 347–364.

STEWARTSON, K. 1982 D'Alembert's Paradox. *SIAM Rev.* **23**, 308–343.

STEWARTSON, K., SMITH, F. T. & KAUPS, K. 1982 On marginal separation. *Stud. Appl. Maths* **67**, 45–61.

VELDMAN, A. E. P. 1979 A numerical method for the calculation of laminar incompressible boundary layers with strong-inviscid interaction. *NLR* TR 79023L.

WILLIAMS, P. G. 1975 A reverse flow computation in the theory of self-induced separation. In *Proc. 4th Intl Conf. on Numerical Methods in Fluid Dynamics* (ed. R. D. Richtmyer). Lecture notes in physics, vol. 35, pp. 445–451.